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2000 J. Phys. A: Math. Gen. 33 3467

(http://iopscience.iop.org/0305-4470/33/17/305)

J. Phys. A: Math. Gen. **33** (2000) 3467–3481. Printed in the UK PII: S0305-4470(00)05738-3

Shape-invariant potentials depending on n parameters transformed by translation

José F Cariñena and Arturo Ramos

Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza, 50009, Zaragoza, Spain

E-mail: `jfc@posta.unizar.es` and `arrg@posta.unizar.es`

Received 6 June 1999, in final form 18 February 2000

Abstract. Shape-invariant potentials in the sense of Gendenshteĭn (1983 *JETP Lett.* **38** 356) which depend on more than two parameters are not know to date. Cooper *et al* (1987 *Phys. Rev.* D **36** 2458) posed the problem of finding a class of shape-invariant potentials which depend on n parameters transformed by translation, but it was not solved. We analyse the problem using some properties of the Riccati equation and find the general solution.

1. Introduction

There has been much interest in the search for exactly solvable problems in quantum mechanics from the early days of the theory to date. In this respect, the factorization method introduced by Schrödinger [16–18] and later developed by Infeld and Hull [12] has been shown to be very efficient. Later, the introduction of supersymmetric quantum mechanics by Witten [20] and the concept of shape invariance by Gendenshteĭn [11] renewed, to a great extent, the interest in the subject. For an excellent review, see [9].

In particular, shape-invariant problems have been shown to be exactly solvable, and it was observed that a number of known exactly solvable potentials belonged to such a class. The natural question which arose was whether all exactly solvable problems have the property of being shape invariant in the sense of [11]. This question was treated in an interesting paper several years ago [8]. There, the Natanzon class of potentials [15] was investigated in detail. Following that line of reasoning, the authors gave a classification of shape-invariant potentials whose parameters are transformed by translation. They proposed the general case which depends on an arbitrary but finite number, n, of parameters, and established the equations to be solved in order to find such a class. However, they asserted to have failed to find any solution of the equations.

For several years this class of shape-invariant potentials has been considered to be a good candidate to enlarge the class of known solutions of the shape-invariance condition, see, for example, [1, 2]. However, the solutions are presently unknown.

As it seems to be an interesting problem we have analysed it carefully and proved that it is possible to find the solution in an easy way. The main point behind our method is to use, in an appropriate way, some interesting properties of a related Riccati equation. As a consequence, the aim of this paper is to answer to the question proposed in [8].

The organization of the paper is as follows. After a quick description of the problem of shape invariance in section 2, in section 3 we will develop the mathematical study of a particularly interesting first-order ordinary differential equation system of key importance for the problem. Then in section 4 we will proceed to study the problem of shape-invariant potentials depending on n parameters. We will perform some *Ansätze* for the superpotentials assuming translations as the transformation law for the parameters, including the one proposed in [8] and its more immediate generalizations. The results are presented in tables 1–4.

2. Shape invariance and the factorization method

We recall some basic ideas of the theory of related operators, the concept of partner potentials and shape invariance. Two Hamiltonians

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) \qquad \tilde{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \tilde{V}(x) \tag{1}$$

are said to be related whether there exists an operator A such that $AH = \tilde{H}A$, where A need not be invertible. If we assume that

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + W(x) \tag{2}$$

then, the relation $AH = \tilde{H}A$ leads to

$$W(V - \tilde{V}) = -W'' - V' \qquad V - \tilde{V} = -2W' \tag{3}$$

while the relation $HA^\dagger = A^\dagger\tilde{H}$ leads to

$$W(V - \tilde{V}) = W'' - \tilde{V}' \qquad V - \tilde{V} = -2W'. \tag{4}$$

One can easily integrate both pairs of equations; we then obtain

$$V = W^2 - W' + c \qquad \tilde{V} = W^2 + W' + d$$

where c and d are constants. But taking into account the equation $V - \tilde{V} = -2W'$ we have $c = d$. Therefore (see, for example, [5]), two Hamiltonians H and $\tilde{H}$ of the form (1) can be related by a first-order differential operator A such as (2) if and only if there exists a real constant d such that W satisfies the pair of Riccati equations

$$V - d = W^2 - W' \qquad \tilde{V} - d = W^2 + W' \tag{5}$$

and then the Hamiltonians can be factorized as

$$H = A^\dagger A + d \qquad \tilde{H} = AA^\dagger + d. \tag{6}$$

Using equations (5) we obtain the equivalent pair

$$\tilde{V} - d = -(V - d) + 2W^2 \qquad \tilde{V} = V + 2W'. \tag{7}$$

The potentials $\tilde{V}$ and V are usually said to be *partners*.

We would like to remark that these equations have an intimate relation with what it is currently known as *Darboux transformations* in the context of one-dimensional or supersymmetric quantum mechanics. In fact, it is easy to prove that the first of equations (5) can be transformed into a Schrödinger equation, $-\phi'' + (V(x) - d)\phi = 0$, by means of the change $-\phi'/\phi = W$, and by means of $\tilde{\phi}'/\tilde{\phi} = W$ the second equation of (5) transforms into $-\tilde{\phi}'' + (\tilde{V}(x) - d)\tilde{\phi} = 0$. The relation between V and $\tilde{V}$ is given by (7). Obviously, $\phi\tilde{\phi} = 1$, up to a non-vanishing constant factor. It is also worth noting that these Schrödinger equations express that ϕ and $\tilde{\phi}$ are respective eigenfunctions of the Hamiltonians (1) for the eigenvalue d. These are the essential points of the mentioned Darboux transformations, as exposed, for example, in [13, pp 7, 24].

The concept of *shape invariance* was introduced by Gendenshteïn [11]: V is assumed to depend on a certain set of parameters, and equations (5) define V and $\tilde{V}$ in terms of a superpotential W. The condition for a partner $\tilde{V}$ to be of the same form as V but for a different choice of the values of the parameters involved in V, is called the shape-invariance condition [11].

More explicitly, if $V = V(x, a)$ and $\tilde{V} = \tilde{V}(x, a)$, where a denotes a set of parameters, Gendenshteïn [11] showed that if we assume the further relation between $V(x, a)$ and $\tilde{V}(x, a)$ given by

$$\tilde{V}(x, a) = V(x, f(a)) + R(f(a)) \tag{8}$$

where f is a transformation of the set of parameters a and $R(f(a))$ is a remainder not depending on x, then the complete spectra of the Hamiltonians H and $\tilde{H}$ can be easily found. Just writing the a-dependence equations (5) become

$$V(x, a) - d = W^2 - W' \qquad \tilde{V}(x, a) - d = W^2 + W'. \tag{9}$$

Therefore, we will assume that $V(x, a)$ and $\tilde{V}(x, a)$ are obtained from a superpotential function $W(x, a)$ by means of

$$V(x, a) - d = W^2(x, a) - W'(x, a) \qquad \tilde{V}(x, a) - d = W^2(x, a) + W'(x, a). \tag{10}$$

The shape-invariance property in the sense of [11] requires the further condition (8) to be satisfied.

The relationship of a slight generalization of the factorization method developed by Infeld and Hull [12] with shape-invariance theory has been explicitly established in [7]. There, the following identifications between the symbols used in the factorization method and those of shape-invariance problems were found:

$$\tilde{V}(x, a) - d = -r(x, f(a)) - L(a) \tag{11}$$

$$V(x, a) - d = -r(x, a) - L(a) \tag{12}$$

$$W(x, a) = k(x, a) \tag{13}$$

$$R(f(a)) = L(f(a)) - L(a). \tag{14}$$

3. General solution of equations $y^2 + y' = a$, $zy + z' = b$

Next we will study the general solution of a specific first-order ordinary differential equation system. It will play a key role in the derivation of the main subject in this paper. The system is

$$y^2 + y' = a \tag{15}$$

$$yz + z' = b \tag{16}$$

where a and b are real constants and the prime denotes the derivative with respect to x. Equation (15) is a Riccati equation with constant coefficients, meanwhile (16) is an inhomogeneous linear first-order differential equation for z, provided the function y is known. The general solution of (16) is easily obtained once we know the solutions of (15), e.g. by means of

$$z(x) = \frac{b\int^x \exp\{\int^\xi y(\eta)\,\mathrm{d}\eta\}\,\mathrm{d}\xi + D}{\exp\{\int^x y(\xi)\,\mathrm{d}\xi\}} \tag{17}$$

where D is an integration constant [7].

The general Riccati equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a_2(x)y^2 + a_1(x)y + a_0(x) \tag{18}$$

where $a_2(x)$, $a_1(x)$ and $a_0(x)$ are differentiable functions of the independent variable x, has very interesting properties. It should be noted that in the most general case there is no way of writing the general solution by using some quadratures, but one can integrate it completely if one particular solution $y_1(x)$ of (18) is known. Then, the change of variable (see, for example, [10, 14])

$$u = \frac{1}{y_1 - y} \qquad \text{with inverse} \quad y = y_1 - \frac{1}{u} \tag{19}$$

transforms (18) into the inhomogeneous first-order linear equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -(2a_2y_1 + a_1)u + a_2 \tag{20}$$

which can be integrated by two quadratures. An alternative change of variable was also proposed recently [6]:

$$u = \frac{yy_1}{y_1 - y} \qquad \text{with inverse} \quad y = \frac{uy_1}{u + y_1}. \tag{21}$$

This change transforms (18) into the inhomogeneous first-order linear equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \left(\frac{2a_0}{y_1} + a_1\right)u + a_0 \tag{22}$$

which is also integrable by two quadratures. We also remark that the general Riccati equation (18) admits the identically vanishing function as a solution if and only if $a_0(x) = 0$ for all x in the domain of the solution.

However, the most important property of Riccati equation is that when three particular solutions of (18), $y_1(x)$, $y_2(x)$, $y_3(x)$ are known, the general solution y can be automatically written, by means of the formula

$$y = \frac{y_2(y_3 - y_1)k + y_1(y_2 - y_3)}{(y_3 - y_1)k + y_2 - y_3} \tag{23}$$

where k is a constant determining each solution. As an example, it is easy to check that $y|_{k=0} = y_1$, $y|_{k=1} = y_3$ and that the solution y_2 is obtained as the limit of k going to ∞. For more information on geometric and group theoretic aspects of the Riccati equation see, for example, [3, 4, 6, 19].

We are interested here in the simpler case of the Riccati equation with constant coefficients (15). The general equation of this type is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a_2y^2 + a_1y + a_0 \tag{24}$$

where a_2, a_1 and a_0 are now real constants, $a_2 \neq 0$. This equation, unlike the general Riccati equation (18), is always integrable by quadratures, and the form of the solutions depends strongly on the sign of the discriminant $\Delta = a_1^2 - 4a_0a_2$. This can be seen by separating the differential equation (24) in the form

$$\frac{\mathrm{d}y}{a_2y^2 + a_1y + a_0} = \frac{\mathrm{d}y}{a_2((y + \frac{a_1}{2a_2})^2 - \frac{\Delta}{4a_2^2})} = \mathrm{d}x.$$

Integrating (24) in this way we obtain non-constant solutions.

Looking for constant solutions of (24) amounts to solving an algebraic second-degree equation. So, if $\Delta > 0$ there will be two different real constant solutions, when $\Delta = 0$ there is only one constant real solution, and if $\Delta < 0$ we have no constant real solutions at all.

These properties may be used for finding the general solution of (15). For this equation the discriminant Δ is just $4a$. Then, if $a > 0$ we can write $a = c^2$, where $c > 0$ is a real number. The non-constant particular solution

$$y_1(x) = c \tanh(c(x - A)) \tag{25}$$

where A is an arbitrary integration constant, is readily found by direct integration. In addition, there exist two different constant real solutions,

$$y_2(x) = c \qquad y_3(x) = -c. \tag{26}$$

The general solution obtained using formula (23), is

$$y(x) = c\frac{B \sinh(c(x - A)) - \cosh(c(x - A))}{B \cosh(c(x - A)) - \sinh(c(x - A))} \tag{27}$$

where $B = (2 - k)/k$, k being the arbitrary constant in (23). Substituting in (17) we obtain the general solution for $z(x)$,

$$z(x) = \frac{\frac{b}{c}\{B \sinh(c(x - A)) - \cosh(c(x - A))\} + D}{B \cosh(c(x - A)) - \sinh(c(x - A))} \tag{28}$$

where D is a new integration constant.

For the case $a = 0$, a particular solution is

$$y_1(x) = \frac{1}{x - A} \tag{29}$$

where A is an integration constant. If we apply the change of variable (21) with y_1 given by (29), then (15) with $a = 0$ transforms into $\mathrm{d}u/\mathrm{d}x = 0$. Then, the general solution for (15) with $a = 0$ is

$$y(x) = \frac{B}{1 + B(x - A)} \tag{30}$$

with A and B being arbitrary integration constants. Substituting in (17) we obtain the general solution for $z(x)$ in this case,

$$z(x) = \frac{b(\frac{B}{2}(x - A)^2 + x - A) + D}{1 + B(x - A)} \tag{31}$$

where D is a new integration constant.

If now $a = -c^2 < 0$, where $c > 0$ is a real number, by direct integration we find the particular solution

$$y_1(x) = -c \tan(c(x - A)) \tag{32}$$

where A is an arbitrary integration constant. With either the change of variable (19) or alternatively (21), with $y_1(x)$ given by (32), we get the general solution of (15) for $a > 0$:

$$y(x) = -c\frac{B \sin(c(x - A)) + \cos(c(x - A))}{B \cos(c(x - A)) - \sin(c(x - A))} \tag{33}$$

where $B = cF$, and F is an arbitrary constant. Substituting in (17) we obtain the general solution for $z(x)$ in this case,

$$z(x) = \frac{\frac{b}{c}\{B \sin(c(x - A)) + \cos(c(x - A))\} + D}{B \cos(c(x - A)) - \sin(c(x - A))} \tag{34}$$

Table 1. General solutions of the equations (15) and (16). A, B and D are integration constants. The constant B selects the particular solution of (15) in each case.

Sign of a	$y(x)$	$z(x)$
$a = c^2 > 0$	$c\dfrac{B\sinh(c(x-A)) - \cosh(c(x-A))}{B\cosh(c(x-A)) - \sinh(c(x-A))}$	$\dfrac{\frac{b}{c}\{B\sinh(c(x-A)) - \cosh(c(x-A))\} + D}{B\cosh(c(x-A)) - \sinh(c(x-A))}$
$a = 0$	$\dfrac{B}{1 + B(x-A)}$	$\dfrac{b(\frac{B}{2}(x-A)^2 + x - A) + D}{1 + B(x-A)}$
$a = -c^2 < 0$	$-c\dfrac{B\sin(c(x-A)) + \cos(c(x-A))}{B\cos(c(x-A)) - \sin(c(x-A))}$	$\dfrac{\frac{b}{c}\{B\sin(c(x-A)) + \cos(c(x-A))\} + D}{B\cos(c(x-A)) - \sin(c(x-A))}$

where D is a new integration constant.

These solutions can be written in many mathematically equivalent ways. We have tried to give their simplest form and in such a way that the symmetry between the solutions for the cases $a > 0$ and $a < 0$ were clearly recognized. Indeed, the general solution of (15) for $a > 0$ can be transformed into that of the case $a < 0$ by means of the formal changes $c \to \mathrm{i}c$, $B \to \mathrm{i}B$ and the identities $\sinh(\mathrm{i}x) = \mathrm{i}\sin(x)$, $\cosh(\mathrm{i}x) = \cos(x)$. The results are summarized in table 1.

Looking at the general solution of (15) for $a > 0$, i.e. equation (27), one could be tempted to write it in the form of a logarithmic derivative,

$$y(x) = \frac{\mathrm{d}}{\mathrm{d}x}\log|B\cosh(c(x-A)) - \sinh(c(x-A))|.$$

This is equivalent except for $B \to \infty$. In fact, if we want to calculate

$$\lim_{B\to\infty}\frac{\mathrm{d}}{\mathrm{d}x}\log|B\cosh(c(x-A)) - \sinh(c(x-A))|$$

we *cannot* interchange the limit with the derivative, otherwise we would get an incorrect result. But this limit for B is particularly important since when taking it in (27), we recover the particular solution (25). A similar thing occurs in the general solutions (30) and (33). When taking the limit $B \to \infty$ we recover, respectively, the particular solutions (29) and (32), from which we have started. Both (30) and (33) can be written in the form of a logarithmic derivative, but then the limit $B \to \infty$ could not be calculated properly.

4. Shape-invariant potentials depending on an arbitrary number of parameters transformed by translation

We will try now to generalize the class of possible factorizations considered in [7, 12]. We analyse the possibility of introducing superpotentials depending on an arbitrary but finite number of parameters n which transforms by translation. In turn, this will give the still-unsolved problem proposed in [8].

More explicitly, suppose that within the parameter space some of them transform according to

$$f(a_i) = a_i - \epsilon_i \qquad \forall i \in \Gamma \tag{35}$$

and the remainder according to

$$f(a_j) = a_j + \epsilon_j \qquad \forall j \in \Gamma' \tag{36}$$

where $\Gamma \cup \Gamma' = \{1, \ldots, n\}$, and $\epsilon_i \neq 0$ for all i. Using a reparametrization, one can normalize each parameter in units of ϵ_i, that is, we can introduce the new parameters

$$m_i = \frac{a_i}{\epsilon_i} \qquad \forall i \in \Gamma \qquad \text{and} \qquad m_j = -\frac{a_j}{\epsilon_j} \qquad \forall j \in \Gamma' \tag{37}$$

for which the transformation law reads, with a slight abuse of the notation f,

$$f(m_i) = m_i - 1 \qquad \forall i = 1, \ldots, n. \tag{38}$$

Note that with these normalizations, the initial values of each m_i are defined by some value in the interval $(0, 1] \pmod{\mathbb{Z}}$.

We will use the notation $m - 1$ for the n-tuple $m - 1 = (m_1 - 1, m_2 - 1, \ldots, m_n - 1)$. The transformation law for the parameters (38) is just a particular case of a more general transformation considered in [7]. As a corollary of a result proved there we have the following one. The problem of finding the square integrable solutions of the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + r(x, m)y + \lambda y = 0 \tag{39}$$

according to the generalization of the Infeld and Hull factorization method treated in [7, section 3], is equivalent to that of solving the discrete eigenvalue problem of shape-invariant potentials in the sense of [11] depending on the same n-tuple of parameters $m \equiv (m_1, m_2, \ldots, m_n)$ which transform according to (38).

In order to find solutions for these problems, we should find solutions of the difference-differential equation

$$k^2(x, m+1) - k^2(x, m) + \frac{\mathrm{d}k(x, m+1)}{\mathrm{d}x} + \frac{\mathrm{d}k(x, m)}{\mathrm{d}x} = L(m) - L(m+1) \tag{40}$$

where now $m = (m_1, m_2, \ldots, m_n)$ denotes the set of parameters and $m + 1$ means $m + 1 = (m_1 + 1, m_2 + 1, \ldots, m_n + 1)$, and $L(m)$ is some function to be determined, related to $R(m)$ by $R(m) = L(m) - L(m + 1)$. Equation (40) is essentially equivalent to the shape-invariance condition $\tilde{V}(x, m) = V(x, m - 1) + R(m - 1)$ for problems defined by (38) [7]. We would like to remark that (40) always has the trivial solution $k(x, m) = h(m)$, for every arbitrary function $h(m)$ of the parameters only.

Our first assumption for the dependence of $k(x, m)$ on x and m will be a generalization of the one used for the case of one parameter introduced in [12]:

$$k(x, m) = k_0(x) + m k_1(x) \tag{41}$$

where k_0 and k_1 are functions of x only. The generalization to n parameters is

$$k(x, m) = g_0(x) + \sum_{i=1}^{n} m_i g_i(x). \tag{42}$$

This form for $k(x, m)$ is exactly the same as the one proposed in [8, equations (6.24)] taking into account (37) and (38), up to a slightly different notation. Substituting into (40) we obtain

$$L(m) - L(m+1) = 2\sum_{j=1}^{n} m_j \left(g_j' + g_j \sum_{i=1}^{n} g_i \right) + \sum_{j=1}^{n} \left(g_j' + g_j \sum_{i=1}^{n} g_i \right) + 2\left(g_0' + g_0 \sum_{i=1}^{n} g_i \right). \tag{43}$$

Since the coefficients of the powers of each m_i have to be constant, we obtain the following first-order differential equation system to be satisfied:

$$g_j' + g_j \sum_{i=1}^{n} g_i = c_j \qquad \forall j \in \{1, \ldots, n\} \tag{44}$$

$$g_0' + g_0 \sum_{i=1}^{n} g_i = c_0 \tag{45}$$

where c_i, $i \in \{0, 1, \ldots, n\}$ are real constants.

The solution of the system can be found by using barycentric coordinates for the g_i, that is, the functions which separate the unknowns g_i in their mass-centre coordinates and relative ones. Hence, we will make the following change of variables and use the notations

$$g_{cm}(x) = \frac{1}{n}\sum_{i=1}^{n} g_i(x) \tag{46}$$

$$v_j(x) = g_j(x) - g_{cm}(x) = \frac{1}{n}\left(ng_j(x) - \sum_{i=1}^{n} g_i(x)\right) \tag{47}$$

$$c_{cm} = \frac{1}{n}\sum_{i=1}^{n} c_i \tag{48}$$

where $j \in \{1, \ldots, n\}$. Note that not all of the functions v_j are now linearly independent, but only $n-1$ since $\sum_{j=1}^{n} v_j = 0$.

Taking the sum of equations (44) we obtain that ng_{cm} satisfies the Riccati equation with constant coefficients

$$ng'_{cm} + (ng_{cm})^2 = nc_{cm}.$$

On the other hand, we will consider the independent functions $v_j(x)$, $j \in \{2, \ldots, n\}$ to complete the system. Using equations (47) and (44) we find

$$\begin{aligned} v'_j &= \frac{1}{n}\left(ng'_j - \sum_{i=1}^{n} g'_i\right) \\ &= \frac{1}{n}(g'_j - g'_1 + g'_j - g'_2 + \cdots + g'_j - g'_j + \cdots + g'_j - g'_n) \\ &= -v_j ng_{cm} + c_j - c_{cm} \end{aligned}$$

and we will take the corresponding equations from 2 to n. The system of equations (44) and (45) is written in the new coordinates as

$$ng'_{cm} + (ng_{cm})^2 = nc_{cm} \tag{49}$$

$$v'_j + v_j ng_{cm} = c_j - c_{cm} \qquad \forall j \in \{2, \ldots, n\} \tag{50}$$

$$g'_0 + g_0 ng_{cm} = c_0 \tag{51}$$

and therefore the motion of the centre of mass is decoupled from the other coordinates. But we already know the general solution of equation (49), which is nothing but equation (15) studied in the preceding section with the identification of y and a with ng_{cm} and nc_{cm}, respectively. Therefore, the possible solutions depend on the sign of nc_{cm}, that is, on the sign of the sum $\sum_{i=1}^{n} c_i$ of all the constants appearing in equations (44). Moreover, all the remaining equations (50) and (51) are linear differential equations of the form (16), identifying z as v_j or g_0, and the constant b as $c_j - c_{cm}$ or c_0, respectively. The general solution of these equations is readily found once ng_{cm} is known, by means of formula (17) adapted to each case. As a result the general solutions for the variables ng_{cm}, v_j and g_0 are found directly by merely consulting with table 1 and making the proper substitutions. The results are shown in table 2.

Once the solutions of equations (49)–(51) are known it is easy to find the expressions for $g_i(x)$ and $g_0(x)$ by reversing the change defined by (46) and (47). It is easy to prove that it is indeed invertible with inverse change given by

$$g_1(x) = g_{cm}(x) - \sum_{i=2}^{n} v_i(x) \tag{52}$$

$$g_j(x) = g_{cm}(x) + v_j(x) \qquad \forall j \in \{2, \ldots, n\}. \tag{53}$$

Table 2. General solutions for the differential equation system (49)–(51). A, B, D_0 and D_j are arbitrary constants. The constant B selects the particular solution of (49) for each sign of nc_{cm}.

Sign of nc_{cm}	$ng_{cm}(x)$	$v_j(x)$ for $j \in \{2, \ldots, n\}$ and $g_0(x)$
$nc_{cm} = C^2 > 0$	$Cf_+(x, A, B, C)$	$\frac{c_j - c_{cm}}{C} f_+(x, A, B, C) + D_j h_+(x, A, B, C)$
		$\frac{c_0}{C} f_+(x, A, B, C) + D_0 h_+(x, A, B, C)$
$nc_{cm} = 0$	$Bf_0(x, A, B)$	$(c_j - c_{cm})h_0(x, A, B) + D_j f_0(x, A, B)$
		$c_0 h_0(x, A, B) + D_0 f_0(x, A, B)$
$nc_{cm} = -C^2 < 0$	$-Cf_-(x, A, B, C)$	$\frac{c_j - c_{cm}}{C} f_-(x, A, B, C) + D_j h_-(x, A, B, C)$
		$\frac{c_0}{C} f_-(x, A, B, C) + D_0 h_-(x, A, B, C)$

where

$$f_+(x, A, B, C) = \frac{B \sinh(C(x - A)) - \cosh(C(x - A))}{B \cosh(C(x - A)) - \sinh(C(x - A))} \quad h_+(x, A, B, C) = \frac{1}{B \cosh(C(x - A)) - \sinh(C(x - A))}$$

$$f_0(x, A, B) = \frac{1}{1 + B(x - A)} \quad h_0(x, A, B) = \frac{\frac{B}{2}(x - A)^2 + x - A}{1 + B(x - A)}$$

$$f_-(x, A, B, C) = \frac{B \sin(C(x - A)) + \cos(C(x - A))}{B \cos(C(x - A)) - \sin(C(x - A))} \quad h_-(x, A, B, C) = \frac{1}{B \cos(C(x - A)) - \sin(C(x - A))}$$

Table 3. General solutions for $k(x, m)$ of the form (42). A, B are arbitrary constants. $\tilde{D}$ denotes the combination $D_0 + \sum_{i=2}^{n} D_i(m_i - m_1)$, where D_0, D_i are the same as in table 2. The constant B selects the particular solution of (49) for each sign of nc_{cm}.

Sign of nc_{cm}	$k(x, m) = g_0(x) + \sum_{i=1}^{n} m_i g_i(x)$
$nc_{cm} = C^2 > 0$	$\frac{1}{C}(c_0 + \sum_{i=1}^{n} m_i c_i) f_+(x, A, B, C) + \tilde{D} h_+(x, A, B, C)$
$nc_{cm} = 0$	$(c_0 + \sum_{i=1}^{n} m_i c_i) h_0(x, A, B) + (\tilde{D} + B \frac{\sum_{i=1}^{n} m_i}{n}) f_0(x, A, B)$
$nc_{cm} = -C^2 < 0$	$\frac{1}{C}(c_0 + \sum_{i=1}^{n} m_i c_i) f_-(x, A, B, C) + \tilde{D} h_-(x, A, B, C)$

where $f_+ = f_+(x, A, B, C)$, $f_0 = f_0(x, A, B)$, $f_- = f_-(x, A, B, C)$
$h_+ = h_+(x, A, B, C)$, $h_0 = h_0(x, A, B)$, $h_- = h_-(x, A, B, C)$ are defined as in table 2

For each of the three families of solutions shown in table 2, one can quickly find the corresponding functions $g_i(x)$, $g_0(x)$, and hence the function $k(x, m)$ according to (42). The results are shown in table 3.

We can now calculate the corresponding shape-invariant partner potentials by means of the formulae (10), (13) and (14) adapted to this case. The results are shown in table 4.

Let us comment on the solutions for the function $k(x, m)$ in table 3 and for the shape-invariant potentials in table 4 we have just found. It is remarkable that the constants c_i, c_0, of equations (44), (45) always appear in the solutions by means of the combination $c_0 + \sum_{i=1}^{n} m_i c_i$. On the other hand, $\tilde{D}$ does not change under the transformation $m_i \to m_i - 1$ since it depends only on differences of the m_i. As $D_0, D_2, \ldots, D_n$ are arbitrary constants, $\tilde{D} = D_0 + \sum_{i=2}^{n} D_i(m_i - m_1)$ can also be regarded as an arbitrary constant. It is very easy to check that the functions $k(x, m)$ indeed satisfy (40), just taking into account that $nc_{cm} = \sum_{i=1}^{n} c_i$ and that when $nc_{cm} = C^2$, $\sum_{i=1}^{n} c_i/C = C$, meanwhile $\sum_{i=1}^{n} c_i/C = -C$ when $nc_{cm} = -C^2$. Obviously, for the case $nc_{cm} = 0$ we have $\sum_{i=1}^{n} c_i = 0$. As we have mentioned already, (40) is essentially equivalent to the shape-invariance condition

Table 4. Shape-invariant partner potentials which depend on n parameters transformed by traslation, when $k(x,m)$ is of the form (42) and $m=(m_1,\ldots,m_n)$. The shape-invariance condition $\tilde V(x,m)=V(x,m-1)+R(m-1)$ is satisfied in each case. A, B and $\tilde D$ are arbitrary constants.

Sign of nc_{cm}	$V(x,m)-d$, $\tilde V(x,m)-d$ and $R(m)$ when $k(x,m)=g_0(x)+\sum_{i=1}^n m_i g_i(x)$
$nc_{cm}=C^2>0$	$\frac{(c_0+\sum_{i=1}^n m_ic_i)^2}{\sum_{i=1}^n c_i}f_+^2+\frac{\tilde D}{C}(2(c_0+\sum_{i=1}^n m_ic_i)+\sum_{i=1}^n c_i)f_+h_+$ $+(\tilde D^2-(B^2-1)(c_0+\sum_{i=1}^n m_ic_i))h_+^2$
	$\frac{(c_0+\sum_{i=1}^n m_ic_i)^2}{\sum_{i=1}^n c_i}f_+^2+\frac{\tilde D}{C}(2(c_0+\sum_{i=1}^n m_ic_i)-\sum_{i=1}^n c_i)f_+h_+$ $+(\tilde D^2+(B^2-1)(c_0+\sum_{i=1}^n m_ic_i))h_+^2$
	$R(m)=L(m)-L(m+1)=2(c_0+\sum_{i=1}^n m_ic_i)+\sum_{i=1}^n c_i$
$nc_{cm}=0$	$(c_0+\sum_{i=1}^n m_ic_i)^2h_0^2+\left(\tilde D+B\frac{\sum_{i=1}^n m_i}{n}\right)\left(\tilde D+B\left(\frac{\sum_{i=1}^n m_i}{n}+1\right)\right)f_0^2$ $+2(c_0+\sum_{i=1}^n m_ic_i)\left(\tilde D+B\left(\frac{\sum_{i=1}^n m_i}{n}+\frac{1}{2}\right)\right)f_0h_0-(c_0+\sum_{i=1}^n m_ic_i)$
	$(c_0+\sum_{i=1}^n m_ic_i)^2h_0^2+\left(\tilde D+B\frac{\sum_{i=1}^n m_i}{n}\right)\left(\tilde D+B\left(\frac{\sum_{i=1}^n m_i}{n}-1\right)\right)f_0^2$ $+2(c_0+\sum_{i=1}^n m_ic_i)\left(\tilde D+B\left(\frac{\sum_{i=1}^n m_i}{n}-\frac{1}{2}\right)\right)f_0h_0+(c_0+\sum_{i=1}^n m_ic_i)$
	$R(m)=L(m)-L(m+1)=2(c_0+\sum_{i=1}^n m_ic_i)$
$nc_{cm}=-C^2<0$	$-\frac{(c_0+\sum_{i=1}^n m_ic_i)^2}{\sum_{i=1}^n c_i}f_-^2+\frac{\tilde D}{C}(2(c_0+\sum_{i=1}^n m_ic_i)+\sum_{i=1}^n c_i)f_-h_-$ $+(\tilde D^2-(B^2+1)(c_0+\sum_{i=1}^n m_ic_i))h_-^2$
	$-\frac{(c_0+\sum_{i=1}^n m_ic_i)^2}{\sum_{i=1}^n c_i}f_-^2+\frac{\tilde D}{C}(2(c_0+\sum_{i=1}^n m_ic_i)-\sum_{i=1}^n c_i)f_-h_-$ $+(\tilde D^2+(B^2+1)(c_0+\sum_{i=1}^n m_ic_i))h_-^2$
	$R(m)=L(m)-L(m+1)=2(c_0+\sum_{i=1}^n m_ic_i)+\sum_{i=1}^n c_i$

where $f_+=f_+(x,A,B,C)$, $f_0=f_0(x,A,B)$, $f_-=f_-(x,A,B,C)$
$h_+=h_+(x,A,B,C)$, $h_0=h_0(x,A,B)$, $h_-=h_-(x,A,B,C)$ are defined as in table 2

$\tilde V(x,m)=V(x,m-1)+R(m-1)$, but this can be checked directly. In order to do so, it may be useful to recall several relations that the functions defined in table 2 satisfy. When $nc_{cm}=C^2$ we have

$$f_+'=C(1-f_+^2)=C(B^2-1)h_+^2 \qquad h_+'=-Cf_+h_+$$

when $nc_{cm}=0$,

$$f_0'=-Bf_0^2 \qquad h_0'=-Bf_0h_0+1$$

and finally when $nc_{cm}=-C^2$,

$$f_-'=C(1+f_-^2)=C(B^2+1)h_-^2 \qquad h_-'=Cf_-h_-$$

where the prime denotes the derivative with respect to x. The arguments of the functions are the same as in the mentioned table and have been left out for simplicity.

When we have only one parameter, that is, $n=1$, one recovers the solutions for $k(x,m)=k_0(x)+mk_1(x)$ shown in the first column of [7, table 6], and the corresponding shape-invariant partner potentials of table 7 in the same reference.

For all cases in table 4, the formal expression of $R(m)$ is exactly the same, but either $\sum_{i=1}^n c_i=nc_{cm}$ have different sign or vanish. Let us now consider the problem of how to determine $L(m)$ from $R(m)$. The method does not provide the expression of $L(m)$ but of $L(m)-L(m+1)$. In fact, there is a freedom in determining this function $L(m)$. Fortunately, for the purposes of quantum mechanics the relevant function is $R(m)$, from which the energy spectrum of shape-invariant potentials in the sense of [11] is calculated [7].

However, let us show how this underdetermination appears. Since

$$R(m) = L(m) - L(m+1) = 2\left(c_0 + \sum_{i=1}^{n} m_i c_i\right) + \sum_{i=1}^{n} c_i \tag{54}$$

is a polynomial in the n parameters m_i, and we have considered only polynomial functions of these quantities so far, $L(m)$ should also be a polynomial. It is of degree two, otherwise a simple calculation would show that the coefficients of terms of degree three or higher must vanish. Consequently, we propose $L(m) = \sum_{i,j=1}^{n} r_{ij} m_i m_j + \sum_{i=1}^{n} s_i m_i + t$, where r_{ij} is symmetric, $r_{ij} = r_{ji}$. Therefore, there are $\frac{1}{2}n(n+1) + n + 1$ constants to be determined. Then, making use of the symmetry of r_{ij} in its indices we obtain

$$L(m) - L(m+1) = -2\sum_{i,j=1}^{n} r_{ij} m_i - \sum_{i,j=1}^{n} r_{ij} - \sum_{i=1}^{n} s_i .$$

Comparing with (54) we find the following conditions to be satisfied:

$$-\sum_{j=1}^{n} r_{ij} = c_i \qquad \forall i \in \{1, \dots, n\} \qquad \text{and} \qquad -\sum_{i=1}^{n} s_i = 2c_0.$$

The first of these equations expresses the problem of finding symmetric matrices of order n whose rows (or columns) sum n given numbers. That is, to solve a linear system of n equations with $\frac{1}{2}n(n+1)$ unknowns. For $n > 1$ the solutions determine an affine space of dimension $\frac{1}{2}n(n+1) - n = \frac{1}{2}n(n-1)$. Moreover, for $n > 1$ the second condition always determine an affine space of dimension $n - 1$. The well known case of $n = 1$ [7, 12] gives a unique solution to both conditions. However, the constant t always remains undetermined.

We will now try to find other generalizations of shape-invariant potentials which depend on n parameters transformed by means of a translation. We should try a generalization using inverse powers of the m_i parameters; we already know that for the case $n = 1$ there appear at least three new families of solutions (see table 6 in [7]). So, we will try a solution of the following type, provided $m_i \neq 0$, for all i,

$$k(x, m) = \sum_{i=1}^{n} \frac{f_i(x)}{m_i} + g_0(x) + \sum_{i=1}^{n} m_i g_i(x). \tag{55}$$

Here, $f_i(x)$, $g_i(x)$ and $g_0(x)$ are functions of x to be determined. Substituting into (40) we obtain, after a little algebra,

$$\begin{aligned} L(m) - L(m+1) = &-\sum_{i,j=1}^{n} \frac{f_i f_j (1 + m_i + m_j)}{m_i(m_i+1)m_j(m_j+1)} - 2g_0 \sum_{i=1}^{n} \frac{f_i}{m_i(m_i+1)} \\ &-2\sum_{i,j=1}^{n} \frac{m_j g_j f_i}{m_i(m_i+1)} + 2\sum_{i,j=1}^{n} \frac{g_j f_i}{m_i+1} + \sum_{i=1}^{n} \frac{2m_i+1}{m_i(m_i+1)} \frac{\mathrm{d}f_i}{\mathrm{d}x} + \cdots \end{aligned}$$

where the dots represent the right-hand side of (43). The coefficients of each of the different dependences on the parameters m_i have to be constant. The term

$$-\sum_{i,j=1}^{n} \frac{f_i f_j (1 + m_i + m_j)}{m_i(m_i+1)m_j(m_j+1)}$$

involves a symmetric expression under the interchange of the indices i and j. As a consequence, we obtain that $f_i f_j = \text{Const.}$ for all i, j. Since i and j run independently the only possibility is that $f_i = \text{Const.}$ for all $i \in \{1, \ldots, n\}$. We will assume that at least one of the f_i is different from zero, otherwise we find ourselves in the already studied case. Then, the term

$$-2g_0 \sum_{i=1}^{n} \frac{f_i}{m_i(m_i+1)}$$

gives us $g_0 = \text{Const.}$ and the term which contains the derivatives of the f_i vanishes. The sum of the terms

$$2\sum_{i,j=1}^{n} \frac{g_j f_i}{m_i+1} - 2\sum_{i,j=1}^{n} \frac{m_j g_j f_i}{m_i(m_i+1)}$$

is only zero for $n = 1$. Then, for $n > 1$ the first term provides us with $\sum_{i=1}^{n} g_i = \text{Const.}$ and the second one, $g_i = \text{Const.}$ for all $i \in \{1, \ldots, n\}$. This is just a particular case of the trivial solution. For $n = 1$, however, we obtain more solutions; this is the case already discussed in [7, 12]. It should be noted that, in general,

$$2\sum_{i,j=1}^{n} \frac{g_j f_i}{m_i+1} - 2\sum_{i,j=1}^{n} \frac{m_j g_j f_i}{m_i(m_i+1)} \neq 2\sum_{i,j=1}^{n} \frac{f_i g_j}{m_i+1}\left(1 - \frac{m_j}{m_i}\right)$$

as one could be tempted to write if one does not take care. Taking the last equation as being valid will lead to incorrect results. As a conclusion we obtain that the trial solution $k(x, m)$ corresponding to that of the case $n = 1$ admits no non-trivial generalization to solutions of the type (55).

It can be shown that if we propose further generalizations to greater degree inverse powers of the parameters m_i, the only solution is also a trivial one. For example, if we try a solution of the type

$$k(x,m) = \sum_{i,j=1}^{n} \frac{h_{ij}(x)}{m_i m_j} + \sum_{i=1}^{n} \frac{f_i(x)}{m_i} + g_0(x) + \sum_{i=1}^{n} m_i g_i(x) \tag{56}$$

where $h_{ij}(x) = h_{ji}(x)$, the only possibility we will obtain is that all involved functions of x have to be constant.

Now we try to generalize (42) to higher positive powers. That is, we will now try a solution of the type

$$k(x,m) = g_0(x) + \sum_{i=1}^{n} m_i g_i(x) + \sum_{i,j=1}^{n} m_i m_j e_{ij}(x). \tag{57}$$

Substituting into (40) we obtain, after several calculations,

$$\begin{aligned}
L(m) - L(m+1) = {}& 4\sum_{i,j,k,l=1}^{n} m_i m_j m_k e_{ij} e_{kl} + 4\sum_{i,j,k,l=1}^{n} m_i e_{ij} m_k (e_{kl} + g_k) \\
&+2\sum_{i,j=1}^{n} m_i m_j \left(\sum_{k,l=1}^{n} (e_{kl} + g_l) e_{ij} + \frac{\mathrm{d}e_{ij}}{\mathrm{d}x}\right) \\
&+4\sum_{i,j=1}^{n} m_i e_{ij} \left(\sum_{k,l=1}^{n} (e_{kl} + g_l) + g_0\right) \\
&+2\sum_{i=1}^{n} m_i \left(g_i \sum_{j,k=1}^{n} (e_{jk} + g_j) + \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=1}^{n} (e_{ik} + g_i)\right) \\
&+\sum_{i,j=1}^{n} (e_{ij} + g_i)\left(\sum_{k,l=1}^{n} (e_{lk} + g_l) + 2g_0\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{i,j=1}^{n} (e_{ij} + g_i) + 2g_0\right).
\end{aligned} \tag{58}$$

As in previous cases, the coefficients of each different type of dependence on the parameters m_i have to be constant. Let us analyse the term of higher degree, i.e. the first term on the right-hand side of (58). Since it contains a completely symmetric sum in the parameters m_i, the dependence on the functions e_{ij} should also be completely symmetric in the corresponding indices. For that reason, we rewrite it as

$$4\sum_{i,j,k,l=1}^{n} m_i m_j m_k e_{ij} e_{kl} = \frac{4}{3}\sum_{i,j,k,l=1}^{n} m_i m_j m_k (e_{ij}e_{kl} + e_{jk}e_{il} + e_{ki}e_{jl})$$

from where one finds the necessary condition

$$\sum_{l=1}^{n}(e_{ij}e_{kl} + e_{jk}e_{il} + e_{ki}e_{jl}) = d_{ijk} \qquad \forall i, j, k \in \{1, \ldots, n\}$$

where d_{ijk} are completely symmetric in their three-indices constants. The number of independent equations of this type is just the number of independent components of a completely symmetric tensor in its three indices, each one running from 1 to n. This number is $\frac{1}{6}n(n+1)(n+2)$. The number of independent variables e_{ij} is $\frac{1}{2}n(n+1)$ from the symmetry on the two indices. Then, the number of unknowns minus the number of equations is

$$\tfrac{1}{2}n(n+1) - \tfrac{1}{6}n(n+1)(n+2) = -\tfrac{1}{6}(n-1)n(n+1).$$

For $n = 1$ the system has the simple solution $e_{11} = \text{Const.}$ For $n > 1$ the system is not compatible and has no solutions apart from the trivial one $e_{ij} = \text{Const.}$ for all i, j. In either of these cases, it is very easy to deduce from the other terms in (58) that all of the remaining functions have to be constant as well, provided that not all of the e_{ij} constants vanish. For higher positive power dependence on the m_i parameters a similar result holds. In fact, let us suppose that the higher-order term in our trial solution is of degree q, $\sum_{i_1,\ldots,i_q=1}^{n} m_{i_1} m_{i_2} \ldots m_{i_q} T_{i_1,\ldots,i_q}$, where $T_{i_1,\ldots,i_q}$ is a completely symmetric tensor in its indices. Then, it is easy to prove that the higher-order term appearing after substitution in (40) is a sum whose general term is of degree $2q - 1$ in the m_i, being completely symmetric under the interchange of these parameters. This sum contains the product of $T_{i_1,\ldots,i_q}$ by itself, but with one index summed. One then has to symmetrize the expression for two T in order to obtain the independent equations to be satisfied, which is equal to the number of independent components of a completely symmetric tensor in its $2q - 1$ indices. This number is $(n + 2(q-1))!/(2q-1)!(n-1)!$. The number of independent unknowns is $(n+q-1)!/q!(n-1)!$. So, the number of unknowns minus the one of equations is

$$\frac{(n+q-1)!}{q!(n-1)!} - \frac{(n+2(q-1))!}{(2q-1)!(n-1)!}.$$

This number always vanishes for $n = 1$, which means that the problem is determined and we obtain that $T_{1,\ldots,1} = \text{Const.}$, in agreement with [12, p 28]. If $n > 1$, one can easily check that for $q > 1$ that number is negative and hence there cannot be other solutions apart from the trivial solution $T_{i_1,\ldots,i_q} = \text{Const.}$ for all $i_1, \ldots, i_q \in \{1, \ldots, n\}$. From the terms of lower degree one should conclude that the only possibility is a particular case of the trivial solution.

5. Conclusions and outlook

Let us comment on the relevance of the more important result of this paper, that is, the fact that we have been able to solve the differential equation system (44) and (45). That problem was posed, but not solved, in a often cited paper by Cooper *et al* [8, pp 2471–2]. They use a slightly different notation but one can identify their formulae (6.24) with our (42) and our procedure by an appropriate redefinition of the parameters taking into account (37) and (35). However, they failed to find any solution to these equations (for $n > 2$), and believed that such a solution could hardly exist.

The conclusion is conceptually of great importance. That is, it has been made clear that an arbitrary but finite number of parameters subject to transformation is not a limitation to the existence of shape-invariant partner potentials, and hence, to the existence of exactly solvable problems in quantum mechanics. This leaves the door open to the possibility to pose and perhaps solve further generalizations. We also have the possibility of englobing particular cases of known shape-invariant partner potentials spread over the extensive literature on the subject (see, for example, [9] and references therein) into one simple but powerful scheme of classification. In this sense, we think the solution we have found here is very important as it completes the excellent work started in [8].

Another conceptual point of great importance is that we have gained much more generality in the solution to the problem by a particularly simple but powerful idea. That is, to consider the general solution of the Riccati equation with constant coefficients which gives all subsequent solutions, rather than particular ones. For doing this the important properties of the Riccati equation have been of great use.

As a byproduct of our present results and that of [7] it is not difficult to see that for $n = 1$ most of the solutions contained in [8, section 6], later reproduced, for example, in [9], are directly related to some results of the classic paper [12], since they are solutions of essentially the same equations.

Acknowledgments

AR thanks the Spanish Ministerio de Educación y Cultura for a FPI grant, research project PB96–0717. Support of the Spanish DGES (PB96–0717) is also acknowledged.

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